This activity comes from the Dynamic Number Project, sponsored by the National Science Foundation DR-K12 program (award #0918733). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

More information on the Dynamic Number Project, and additional Geometric Functions activities, are available here: www.kcptech.com/dynamicnumber

## GEOMETRIC FUNCTIONS

This is one of a series of activities in which students explore functions geometrically. Because both the independent and dependent variables of a geometric function are points, students can vary the independent variable by dragging it and observe directly the behavior of the dependent variable. By tracing both variable points, the relationship between them becomes visible as an image on the screen. Students explore domain, range, composition, and inverse through direct manipulation of variables, and these concepts manifest themselves through visual images that reveal their fundamental aspects.

Many students may have met functions only in the numeric realm, and may not be aware as they begin these activities that *function*, *transformation*, and *mapping* are mathematical synonyms. It's strongly recommended that students begin with these two activities (Indentify Functions and Indentify Function Families) to begin to get comfortable with functions in which the variables are points, developing some practice in manipulating the independent variable while observing the behavior of the dependent variable. Class discussions of the similarities and differences between algebraic and geometric functions help students see that these geometric relationships really do behave as functions, and encourage students' awareness of the continuous variability of both geometric and numeric variables. Students come to understand how functions in both realms express relationships between independent and dependent variables.

By engaging with geometric functions and discussing their similarities and differences with numeric functions, students are spurred to go beyond initial simplistic ideas, such as the common belief the essential nature of a function is an algebraic representation in the form $y = 2x$. As they use geometric functions to study domain, range, composition, and related topics, students have an opportunity to work with geometric representations of these concepts, to relate them to algebraic representations, and thus to develop a deeper and more sophisticated understanding of functions.

## ACTIVITY OVERVIEW

This activity introduces the idea of families of functions, using geometric points as the independent and dependent variables. On each page students explore four functions, three of which are from the same family and one of which is from a different family.

 

By exploring the behavior of these functions, students become familiar with functions that use translation, rotation, dilation, reflection, and glide reflection, and they develop their ability to distinguish the behavior of one function from another. This activity prepares students to create their own functions in the next activity, and it prepares them to think about families of numeric functions. By classifying functions based on their behavior, students develop a way of thinking about function families based on the nature of the relationship between the input and output, rather concentrating on representation-dependent features such as the form of an algebraic function's symbolic representation.

Students are encouraged to use a number of different features to compare the behaviors of functions:

- the relationship between shapes traced by the two variables
- the relative directions in which the variables move
- the relative speeds of the variables
- the presence and nature of fixed points

## OBJECTIVES

After completing this activity, students should be able to:

- Drag independent variables and observe and describe the behavior of dependent variables. (Use the drag test.)
- Distinguish between functions based on the shapes traced by the variables.
- Distinguish between functions based on relative speed and direction of the variables.
- Distinguish between functions based on the presence and nature of fixed points.
- Describe several function families (such as translations, rotations, dilations, reflections, and glide reflections) based on the similarity of their behaviors.
- Formulate a working definition of a function family.
- Determine whether a function belongs to a particular family.
- Use function notation appropriately to refer to a function and to a dependent variable.
- Construct new functions belonging to particular families (optional).

Many of these objectives will appear again when students learn about families of numeric functions. In particular, they'll find it useful to attend to relative speed and direction (related rates) and to fixed points of numeric functions.

**Identifying and describing function families:** The exploration and discussion in which students engage to identify and describe function families is the heart of this activity. This process helps them to refine their concept of function, to observe function behavior carefully, and to compare two functions by picking out features that they share, and features that differ. This analysis of the similarities and differences of function behavior, and of the classification into families based on similarities and differences, is the critical goal of this activity.

**On your own:** Some students will enjoy constructing their own functions that belong to different families, and challenging their classmates to investigate their functions and figure out to what families they belong.

This will be easier if your students already have some experience with creating Sketchpad constructions, but page 12 gives fairly explicit step-by-step instructions that many students will be able to follow, even without extensive construction experience. Later activities will focus on such constructions.

## VOCABULARY

***Independent variable*** and ***dependent variable:*** The activity uses these terms rather than *independent point* and *dependent point*. By using the same vocabulary for geometric functions and numeric functions, students are encouraged to see the common nature of these functions, and they can use the same language to state their observations, definitions, and abstractions.

***Input variable*** and ***output variable:*** These less formal terms are introduced, without mentioning them, on page 2 of the worksheet. The even simpler terms *input* and *output* are useful, both for the ease of describing behavior and for the connections to previous experiences in which students may have heard and used these terms.

***Function:*** As they identify function families, students solidify their understanding of the nature of a function.

***Function Family:*** This term comes to have a real meaning to students as they observe the common features of members of a family, and the differences between one family and another.

***Function Notation:*** Though this phrase is not used, students are introduced to the notation, beginning the process of familiarizing themselves with it and using it to talk about functions. The main advantage of using function notation in this activity is to encourage students to think of the function as a named entity, worthy of study in its own right. The objects of study in this activity are not only the variables (which have names), but also the relationship between the variables (the function, which has its own name).

 

# INTRODUCE AND MODEL

Use a projector to show sketch page 1, and ask a student to read the directions. Call another student to the interactive whiteboard (if you have one) or to the computer to drag each of the points on page 1. Ask the class to describe what they observe as some points are dragged and others cannot be selected. They should identify the points as independent and dependent variables, and each pair of points as a function. Encourage them to observe that the color and the notation are hints that make it easy to tell which variables are related, and which of each pair of variables is independent and which is dependent. Tell students to write answers to worksheet questions Q1 and Q2 in their own words.

Draw students' attention to the table in Q3, and tell them that they'll fill in the first row of Q3 as a class before they start working in pairs. Ask them what variable they should drag to investigate function $f$. Discuss briefly the notation used here: the variables are called $a$ and $f(a)$, and the function is called $f$. Try to elicit (or suggest if necessary) the idea that $f(a)$ means the output that results when input variable $a$ has been acted on by function $f$. [Though it's not uncommon to see a function referred to as $f(a)$, it's best at this stage not to conflate the name of the function with the name of the dependent variable.]

Have the student at the computer drag variable $a$, and ask students to describe the behavior of this function. It's important to use questioning techniques that don't lead students to particular answers, that encourage other students to paraphrase each other, and that encourage students to build on each other's descriptions. As students describe various aspects of the function's behavior, encourage them to observe and describe several features that will be important when they describe the behavior of symbolic/numeric functions:

- the *relative directions* in which the variables move,
- the *relative speeds* with which they move, and
- whether (and if so, where) the function has *fixed points.* (A *fixed point* is a location at which the independent and dependent variables coincide.)

To encourage the connections that you'll make later, be specific in using and drawing attention to the italicized terms for these features.

Try to let students themselves suggest trying the *Tracing On* and *Tracing Off* buttons to see whether they are useful. Once students have turned traces on, you may want to ask them to think about how the traces might help to reveal relative direction, relative speed, and fixed points.

Following this discussion, ask students to fill in the first row of the table, using their own words and writing complete sentences.

## EXPLORE

Assign students to partners, send them in pairs to the computers, and ask them to finish filling in the table for Q3, and then work through pages 2 through 11. For each page they should describe, in complete sentences, the behavior of the functions that are similar and the behavior of the function that's different.

Tell students that pages 12 through 14 are optional, but are a nice challenge if they finish the earlier pages.

As you circulate, make sure students are writing clear descriptions, in complete sentences, of both the similarities and differences they observe. As you look at the descriptions, note which ones will be useful for the class discussion — particularly descriptions that will be useful in naming the different families. Encourage students to drag purposefully, and observe students' dragging strategies for figuring out the function that's different.

As you circulate, identify particularly interesting or revealing student observations and strategies. Later, during the discussion, you can call on those students to describe what they noticed or what strategies they used. (For instance, a students might drag all four independent variables at the same time, to compare how the dependent variables move as she moves the mouse.)

## DISCUSS AND SUMMARIZE

When students have had enough time to investigate and describe the examples and non-examples on pages 1 through 11, call them together for a class discussion. (Always have monitors turned off or laptop lids closed for such discussions.)

Begin by asking students which pages they found most interesting. Have a volunteer come to the computer and demonstrate a page that they found interesting. Have them demonstrate, and describe in their own words, the similarities that they observed in three of the functions on that page. Then suggest that the class invent a name of the family being demonstrated. Suitably descriptive names might be *slide*, *follow*, or *keep pace* for a translation, *turn*, *twist*, *circle*, or *spiral* for a rotation, *stretch*, *shrink*, or *scale* for a dilation, *flip*, *reflect*, or *mirror* for a reflection, and *flip-and-slide* or *step* for a glide reflection. (Though *spiral* doesn't describe the behavior of a rotation, it does describe the trajectories when students move the independent variable toward the dependent variable.)

Ask students what strategies they used to find the function that's different. How much did it help to use the traces? What did they notice about how some traces were different from others?

If there's time, continue having students demonstrate enough different pages to look at and invent a class name for at least the translation, rotation, dilation, and reflection families. For each such family, list the features that students identify. (For instance, the relative rates are the same for all but dilation. Direction of the motion of both variables is the same for translation and dilation, always different for rotation, and sometimes different for reflection. Rotation and dilation have a single fixed point, reflection has a line as a fixed point, and translations don't have a fixed point at all.)

Pages 9, 10, and 11 use pictures as the input and output to the various functions. Ask students how they found these pages different from the others. Students may say that it was easier to analyze the pictures; question them as to why they found it easier. What extra information did the pictures give them?

Then ask if these pages used any of the same families that they just named and described, and ask students what other connections they saw between the pages with variables and the pages with pictures. What they would say about the independent and dependent variables on these pages? (There are several ways to answer this question; its main value is to provoke discussion. One answer is to say that the pictures have no single independent and dependent variables. Instead, the entire collection of points that make up the picture has been transformed, with each point being transformed in a similar way to the way in which students transformed multiple values of the independent variable when they dragged and traced. Another answer is to say that the entire picture is an independent variable. This approach gets into complications; you may want to observe that many questions yet to come will be easier to think about if the variables are simple points or numbers, rather than complex pictures or entire sets of numbers.)

As they discuss the picture pages, students may observe that the transformed pictures are similar to the trace of a dependent variable when they dragged the independent variable to make a particular shape.

## ANSWERS

The answers below are summaries, or representative answers. Student answers will vary considerably.

**Q1** The independent variable has a single letter label, and the dependent variable has one letter and then another in parentheses.

**Q2** The label of the independent variable appears inside the parentheses for the related dependent variable.

**Q3** For *f, h,* and *j*, the variables move at the same speed. Depending on the direction, they may move toward each other, along with each other, or away from each other. For *g*, they move in the same direction, but the dependent variable moves more slowly. (Functions *f, h,* and *j* are reflections, and *g* is a dilation.)

**Q4** For *f, g*, and *h*, the variables always move at the same speed and direction. For *j*, they move in the same direction, but the dependent variable moves faster. (Functions *f, g*, and *h* are translations, and function *j* is a dilation.)

**Q5** For *f*, the fixed points lie on a line halfway between the initial locations of the variables. For *g, h*, and *j*, there's a single fixed point at the center of the turn or rotation. (Functions *g, h*, and *j* are rotations, and function *f* is a reflection.)

**Q6** For *p*, the dependent variable always moves at exactly the same speed and direction as the independent variable. For *q, r*, and *s*, the dependent variable moves in the opposite direction when you drag the input toward or away from the output, and in the same direction when you drag the input sideways relative to the output. (Functions *q, r*, and *s* are reflections, and function *p* is a translation.)

**Q7** For *f, g*, and *h*, the variables always move at the same speed but in different directions. For *f* and *g*, the directions are only slightly different; for *h*, they are very different — approximately perpendicular. For *j*, the variables move in the same direction, but the dependent variable moves slightly faster. (Functions *f, g*, and *h* are rotations, and function *j* is a dilation.)

**Q8** Pages 6, 7, and 8:

| Page | Fn | Describe the difference |
|---|---|---|
| 6 | *g* | The input can't ever reach the output. The other functions are reflections, but *g* is like a reflection, but offset. (This is a glide reflection.) |
| 7 | *r* | The other functions are rotations. But for function *r*, the output always stays the same distance and direction from the input. (This is a translation.) |
| 8 | *g* | The output is always the same turn counter-clockwise from the input. (This is a rotation.) The other functions have more rotation the farther they get from the moon. |

**Q9** Pages 9, 10, and 11:

| Page | Different Picture | Describe the difference |
|---|---|---|
| 9 | *h* | Function *h* is a translation; the others are all reflections. |
| 10 | *gazelle* | The gazelle is smaller and pointed the same direction (dilation). The other animals are all turned (rotations). |
| 11 | *butterfly* | The butterfly is reflected; you can move the input and output together. The others are translated as well as reflected (glide reflections). |