Exploring Properties of Operations

Activity Notes

Objective: Students use arithmetic machines to explore various properties of the four fundamental arithmetic operations. They manipulate the machines by dragging two variables, *a* and *b*, observe the results of various calculations, and draw conclusions that reveal similarities and differences between the operations.

Student Audience: Pre-algebra/Algebra 1

Prerequisites: Students should have completed one of the other arithmetic machine activities. Experience with custom tools is helpful. The Present section requires students to create buttons.

Sketchpad Level: Intermediate. Students manipulate a pre-made sketch and use custom tools.

Activity Time: Will vary widely depending on the approach (see General Notes). It would take most students, working alone, much longer than one class period to fill in charts for all nine descriptions. By working in groups, students can complete the activity in one class period. Be sure to allow time for discussion and summarizing.

Setting: Paired Activity, or divide the class into groups of three or four (use **Operation Properties.gsp** in either setting)

GENERAL NOTES

This activity challenges students to analyze and understand the four fundamental arithmetic operations—addition, subtraction, multiplication, and division—from a visual perspective.

It's possible to answer any of the questions in this activity without using the arithmetic machines at all. But that defeats its purpose: Watching the machines in action as you drag pointers *a* and *b* can be fascinating. The four arithmetic operations, normally computed on a discrete, case-by-case basis, yield new insights when viewed from a continuous, motion-based perspective.

To get the most out of this activity, students should not treat the process of finding the answers as a purely mechanical one, dragging the pointers around without thinking about the meaning behind the various arrangements. Encourage students to discuss and reflect as they work.

Group Work: Dividing the class into groups will make the work more manageable and allow time for reflection. Each group can be responsible for filling out the four charts for addition, subtraction, multiplication, and division. Encourage group members to work together, observing each other's experiments and checking each other's work as they fill in the charts together, rather than having individual members work in isolation on specific charts.

Translating Between Representations: Doing this investigation requires students to translate between abstract algebraic statements and descriptions of concrete geometric behavior. First they must translate from the algebraic language in the chart ("$a \otimes b > a$ and $a \otimes b > b$") to the geometric behavior they are looking for ("The $a \otimes b$ marker is to the right of both *a* and *b*"). Once they have investigated the behavior of the model, they must then translate the geometric behavior they have observed back into algebraic language. This translation process is not easy for students, and it may be helpful to address the process explicitly, by having them fill out (perhaps as a class) the Geometric Descriptions of Algebraic Properties chart provided. Sample answers are provided following the Properties charts on the next pages.

An Alternate Approach: One possible variation to the approach suggested in the activity is to focus on one property for all four operations. For example, students might focus on the first property ($a \otimes b = 0$) and fill out the top row of each of the four charts—for addition, subtraction, multiplication, and division.

Additional Investigations: The questions in this activity are the tip of the iceberg when it comes to exploring the arithmetic machines. Why not challenge your students to write and share some of their own questions? Here are a few to consider:

- Describe the behavior of $a \div b$ as *b* is dragged back and forth through 0.
- If $a \cdot b$ equals 0, does knowing the value of *a* allow you to determine a unique value for *b*?
- If $a \div b$ equals 0, does knowing the value of *a* allow you to determine a unique value for *b*?
- When is $a - b$ greater (or less) than $a + b$?
- When is $a \cdot b$ greater (or less) than $a \div b$?

© 2006 Key Curriculum Press

- Under what circumstances does $a + b = a - b$ regardless of where you drag a?
- Under what circumstances does $a \cdot b = a \div b$ regardless of where you drag a?

INVESTIGATE

Following are sample answers for each of the four charts. The specific examples and the "When is it true?" description will vary; many correct responses are possible.

Addition Properties

$a + b = 0$	$a=3$, $a=-2$, $a=0$,	$b=-3$ $b=2$ $b=0$	The sum of two numbers is zero when the numbers are the opposites of each other (or both equal zero).
$a + b = 1$	$a=0$, $a=3$, $a=-3$,	$b=1$ $b=-2$ $b=4$	The sum of two numbers is one when the second number is one more than the opposite of the first.
$a = b = a + b$	$a=0$,	$b=0$	The only way two numbers can both be equal to their sum is when both numbers are zero.
$a = a + b$	$a=5$, $a=-3$, $a=0$,	$b=0$ $b=0$ $b=0$	The sum of two numbers is equal to the first number only if the second number is zero.
$a > 0$, $b > 0$, and $a + b > 0$	$a=1$, $a=0.5$, $a=5$,	$b=1$ $b=1$ $b=10$	When both numbers are positive, their sum is always positive.
$a < 0$, $b < 0$, and $a + b < 0$	$a=-1$, $a=-0.5$, $a=-2$,	$b=-1$ $b=-1$ $b=-3$	When both numbers are negative, their sum is always negative.
$a + b > a$ and $a + b > b$	$a=1$, $a=0.1$, $a=3$,	$b=1$ $b=2$ $b=4$	The sum is greater than either number provided both numbers are positive.
$a + b$ is between a and b	$a=1$, $a=-1$, $a=-5$,	$b=-2$ $b=0.1$ $b=4$	If one number is positive and the other negative, the sum is between the two numbers.
$a + b < a$ and $a + b < b$	$a=-1$, $a=-1$, $a=-10$,	$b=-1$ $b=-10$ $b=-1$	The sum is less than either number if both numbers are negative.

Subtraction Properties

$a - b = 0$	$a=3$, $a=-2$, $a=0$,	$b=3$ $b=-2$ $b=0$	The difference of two numbers is zero when the numbers are equal.
$a - b = 1$	$a=2$, $a=-1$, $a=1$,	$b=1$ $b=-2$ $b=0$	The difference of two numbers is one when the first number is one more than the second.
$a = b = a - b$	$a=0$,	$b=0$	The only way two numbers can both be equal to their difference is when both numbers are zero.
$a = a - b$	$a=5$, $a=-3$, $a=0$,	$b=0$ $b=0$ $b=0$	The difference of two numbers is equal to the first number only if the second number is zero.
$a > 0$, $b > 0$, and $a - b > 0$	$a=1$, $a=1.1$, $a=5$,	$b=0.5$ $b=1$ $b=4$	When both numbers are positive, their difference is positive if the first is greater than the second.
$a < 0$, $b < 0$, and $a - b < 0$	$a=-2$, $a=-1.5$, $a=-4$,	$b=-1$ $b=-1$ $b=-3$	When both numbers are negative, their difference is negative if the first number is less than the second.
$a - b > a$ and $a - b > b$	$a=-1$, $a=-3$, $a=3$,	$b=-1$ $b=-2$ $b=-4$	The difference is greater than either number as long as the second number is negative and the first is greater than twice the second.
$a - b$ is between a and b	$a=5$, $a=-9$, $a=13$,	$b=2$ $b=-4$ $b=6$	If the second number is positive, the difference is between the two numbers when the first is greater than twice the second. If the second number is negative, the difference is between the two numbers when the first is less than twice the second.
$a - b < a$ and $a - b < b$	$a=1$, $a=3$, $a=7$,	$b=1$ $b=2$ $b=4$	The difference is less than either number if the second number is positive and the first is less than twice the second.

© 2006 Key Curriculum Press

Multiplication Properties

$a \cdot b = 0$	$a=0$, $a=-2$, $a=0$,	$b=-3$ $b=0$ $b=0$	The product of two numbers is zero when at least one of the numbers is zero.
$a \cdot b = 1$	$a=1$, $a=3$, $a=1/4$,	$b=1$ $b=1/3$ $b=4$	The product of two numbers is one when the numbers are reciprocals.
$a = b = a \cdot b$	$a=0$, $a=1$,	$b=0$ $b=1$	The only way two numbers can both be equal to their product is when both numbers are zero or both numbers are one.
$a = a \cdot b$	$a=5$, $a=-3$, $a=0$,	$b=1$ $b=1$ $b=1$	The product of two numbers is equal to the first number only if the second number is one.
$a > 0$, $b > 0$, and $a \cdot b > 0$	$a=1$, $a=0.5$, $a=5$,	$b=1$ $b=1$ $b=10$	The product of two positive numbers is positive.
$a < 0$, $b < 0$, and $a \cdot b < 0$	Never		The product of two negative numbers is never negative.
$a \cdot b > a$ and $a \cdot b > b$	$a=1.1$, $a=-1$, $a=3$,	$b=1.1$ $b=-2$ $b=4$	The product is greater than either number either when both numbers are negative or when both numbers are greater than one.
$a \cdot b$ is between a and b	$a=1/2$, $a=1/2$, $a=4$, $a=-3$,	$b=-2$ $b=3$ $b=1/4$ $b=1/4$	The product is between the two numbers if one number is between zero and one and the other is either negative or greater than one.
$a \cdot b < a$ and $a \cdot b < b$	$a=0.5$, $a=-2$, $a=2$,	$b=0.2$ $b=1.5$ $b=-0.1$	The product is less than either number if both numbers are between zero and one, or if one number is negative and the other is greater than one.

Division Properties

$a \div b = 0$	$a=0$, $a=0$, $a=0$,	$b=-3$ $b=2$ $b=1/2$	The quotient of two numbers is zero when the numerator is zero and the denominator is not zero.
$a \div b = 1$	$a=1$, $a=-2$, $a=4$,	$b=1$ $b=-2$ $b=4$	The quotient of two numbers is one when the numbers are equal but not zero.
$a = b = a \div b$	$a=1$,	$b=1$	The only way two numbers can both be equal to their quotient is when both numbers are one.
$a = a \div b$	$a=5$, $a=-3$, $a=0$,	$b=1$ $b=1$ $b=1$	The quotient of two numbers is equal to the numerator only if the denominator is one.
$a > 0$, $b > 0$, and $a \div b > 0$	$a=1$, $a=0.5$, $a=5$,	$b=1$ $b=1$ $b=10$	The quotient of two positive numbers is always positive.
$a < 0$, $b < 0$, and $a \div b < 0$	Never		The quotient of two negative numbers is never negative.
$a \div b > a$ and $a \div b > b$	$a=-1$, $a=0.2$, $a=0.2$,	$b=-2$ $b=0.5$ $b=0.1$	The quotient is greater than either number if both numbers are negative, or if the denominator is between zero and one and the numerator is greater than the square of the denominator.
$a \div b$ is between a and b	$a=5$, $a=1/8$, $a=8$,	$b=2$ $b=1/2$ $b=-4$	The quotient is between the two numbers if the denominator is greater than one and the numerator is greater than the square of the denominator, or if the denominator is less than one and the numerator is between zero and the square of the denominator.
$a \div b < a$ and $a \div b < b$	$a=5$, $a=-1$, $a=3$,	$b=-2$ $b=1/2$ $b=2$	The quotient is less than either number if (a) the denominator is less than zero and the numerator is greater than the square of the denominator, or (b) the denominator is between zero and one and the numerator is less than zero, or (c) the denominator is greater than one and the numerator is between zero and the square of the denominator.

© 2006 Key Curriculum Press

Geometric Descriptions of Algebraic Properties

If you have students fill in this chart, here are sample geometric descriptions of the algebraic properties:

Row	Algebraic Property	Geometric Description
1	$a \otimes b = 0$	The $a \otimes b$ marker points at 0.
2	$a \otimes b = 1$	The $a \otimes b$ marker points at 1.
3	$a = b = a \otimes b$	The *a*, *b*, and $a \otimes b$ markers all point to the same location.
4	$a = a \otimes b$	The *a* and $a \otimes b$ markers point to the same location.
5	$a > 0, b > 0$, and $a \otimes b > 0$	The *a*, *b*, and $a \otimes b$ markers are all to the right of 0.
6	$a < 0, b < 0$, and $a \otimes b < 0$	The *a*, *b*, and $a \otimes b$ markers are all to the left of 0.
7	$a \otimes b > a$ and $a \otimes b > b$	The $a \otimes b$ marker is to the right of both the *a* and the *b* markers.
8	$a \otimes b$ is between *a* and *b*	The $a \otimes b$ marker is between *a* and *b*. (Note that the algebraic description given here isn't very algebraic. The real algebraic description would be $a < a \otimes b < b$ or $b < a \otimes b < a$.)
9	$a \otimes b < a$ and $a \otimes b < b$	The $a \otimes b$ marker is to the left of both *a* and *b*.

© 2006 Key Curriculum Press